MATHEMATICS

Mob. : 9470844028
9546359990

AIM POINTS
MATHEMATICS
DIR. FIROZ AHMAD

Ram Rajya More, Siwan

XIth, XIIth, TARGET IIT-JEE (MAIN + ADVANCE) & COMPETITIVE EXAM. FOR XII (PQRS)

ALGEBRA OF MATRICES

& Their Properties

CONTENTS

Key Concept-I ....

Exercise-I ....

Exercise-II ....

Exercise-III ....

Solutions of Exercise

Page ....

THINGS TO REMEMBER

1. A set of mn numbers (real or imaginary) arranged in the form of a rectanble array of m rows and n columns is called an $m \times n$ matrix.
2. A matrix having only one row is called a row matix.
3. A matrix having only one column is called a column matrix.
4. A matrix in which the number of rows is equal to the number of columns, say n, is called a square matrix of order n.
5. The elements a_{ij} of a square matrix $A = [a_{ij}]_{m\times n}$ for which $i = j$, i.e., the elements $a_{11}, a_{22}, ..., a_{nm}$ are called the diagonal element sand the line along which they lie is called the principal diagonal or leading diagonal.
6. A swuare matrix $A = [a_{ij}]_{n\times n}$ is called a diagonal matrix if all the elements, except those in the leading diagonal,are zero.
   i.e., $a_{ij} = 0$ for $i \neq j$.
7. A square matrix $A = [a_{ij}]_{n\times n}$ is called a scalar matrix, if
   (i) $a_{ij} = 0$ for all $i \neq j$.
   and, (ii) $a_{ij} = c$ for all i, where $c \neq 0$.
8. A square matrix $A = [a_{aj}]_{n\times n}$ is called an identity or a unit matrix, if
   (i) $a_{ij} = 0$ for all $i \neq j$
   and, (ii) $a_{ii} = 1$ for all i.
9. A matrix whose all elements are zero is called a null matrix or a zero matrix.
10. A square matrix $A = [a_{ij}]$ is called
    (i) an upper triangular matrix, if $a_{ij} = 0$ for all $i > j$
    (ii) a lower triangular matrix, if $a_{ij} = 0$ for all $i < j$.
11. Two matrices $A = [a_{ij}]_{m\times n}$ of the same order are equal, if
    $a_{ij} = b_{ij}$ for all $i = 1, 2, ..., k; j = 1, 2, ..., n$
12. If $A = [a_{ij}]_{m\times n}$ are two matrices of the same order $m \times n$, then their sum $A + B$ is an $m \times n$ matrix such that
    $(A + B)_{ij} = a_{ij} + b_{ij}$ for $i = 1, 2, ..., m$ and $j = 1, 2, 3, ..., n$
13. Following are the properties of matrix addition :
    (i) *Commutativity :* If A and B are two matrices of the same order, then
    $A + B = B + A$
    (ii) *Associativity :* If A, B, C are three matrices of the same order, then
    $(A + B) + C = A + (B + C)$
    (iii) *Existence of Identity :* The null matrix is the identity element for matrix addition
    i.e., $A + O = A + O \times A$
    (iv) *Existence of Inverse :* For ev ery matrix $A = [a_{ij}]_{m\times n}$ there exists a matrix $-A = [-a_{ij}]_{m\times n}$ such that
    $A + (-A)\ O = (-A) + A$
    (v) *Cancellation Laws :* If A, B, C are three matrices of the same order, then
    $A + B = A + C \Rightarrow B = C$
    and, $B + A = C + A \Rightarrow B = C$

13. Let $A = [a_{ij}]$ be an $m \times n$ matrix and k be any number called a scalar. Then, the matrix obtained by multiplying every element of A by k is called the scalar multiple of A by k and is denoted by kA. Thus, $kA = [k\, a_{ij}]_{m\times n}$.
Following are the properties of scalar multiplication :
(i) $k(A + B) = kA + kB$
(ii) $(k + l)A = kA + lA$
(iii) $(kl)A = k(lA) = l(kA)$
(iv) $(k)A = -(kA) = k(-A)$
(v) $1A = A$
(vi) $(-1)A = -A$
14. If A and B are two matrices of the same order, then $A - B = A + (-b)$.
15. Two matrices A and B are conformable for the product AB if the number of columns in A is same as the number of rows in B.
If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ are two matrices, then AB is an $m \times p$ such that

$$(AB)_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}$$

Matrix multiplication has the following properties :
(i) Matrix multiplication is not communicative.
(ii) Matrix multiplication is associative i.e., $(AB)\, C = A\, (BC)$ where both sides of the equality are defined.
(iii) Matrix multiplication is distributive over matrix addition
i.e., $A(B + C) = AB + AC$
and, $(B + C)\, A = B\, A + CA$
wherever both sides of the equality are defined.
(iv) If A is an $m \times n$ matrix $I_m\, A = A = A\, I_n$
(v) If A is an $m \times n$ matrix and O is a null matrix, then
$A_{m\times n}\, O_{n\times p} = O_{m\times p}$
and, $O_{p\times m} \times A_{m\times n} = O_{p\times n}$
i.e., the product of a matrix with a null matrix is a null matrix.
16. If A is a square matrix, then we define $A^1 = A$ and $A^{n+1} = A^n \,.\, A$
17. If A is a square matrix and $a_0, a_1, ..., a_n$ are constants, then
$a_0A^n + a_1A^{n-1} + a_2A^{n-1} + ... + a_{n-1}A + a_n$ is called a matrix polynomial.
18. Let $A = [a_{ij}]$ be an $m \times n$ matrix. Then, the transpose of A, denoted by A^T, is an $n \times m$ matrix such that

$(A^T)_{ij} = a_{ij}$ for all $i = 1, 2, m; j = 1, 2, ..., n$.

Following are the properties of transpose of a matrix :
(i) $(A^T)^T = A$
(ii) $(A + B)^T = A^T + B^T$
(iii) $(kA)^T = kA^T)$
(iv) $(AB)^T = B^TA^T$
(v) $(ABC)^T = C^T\, B^T\, A^T$
19. A square matrix $A = [a_{ij}]$ is called a symmetric matrix, if
$a_{ij} = a_{ij}$ for all $i, j \Leftrightarrow A = A^T$
20. A square matrix $A = [a_{ij}]$ is called a skew–symmetric matrix, if
$a_{ij} = a_{ij}$ for all $i, j \Leftrightarrow A^T = -A$
21. Al l main diagonal elements of a skew–symmetric matrix are zero.

22. Every square matrix can be uniquely expressed as the sum of a symmetric and a skew–symmetric matrix.
23. All positive integral powers of a symmetric matrix are symmetric.
24. All odd positive integral powers of a skew–symmetric matrix are skew–symmetric.

EXERCISE-1

1. Construct a 3×4 matrix $A = [a_{aj}]$ whose elements aregiven by
   (i) $a_{ij} = i + j$ (ii) $a_{ij} = i - j$
2. If $\begin{bmatrix} x-y & 2x+z \\ 2x-y & 3z+w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$ find x, y, z, w.
3. A matrix has 12 elements. What are the possible orders it can have ?
4. Construct a 2×2 matrix $A = [a_{ij}]$ whose elements are given by $a_{ij} = \frac{(i+2j)^2}{2}$
5. Find x, y, z and to such that $\begin{bmatrix} x-y & 2z+w \\ 2x-y & 2x+w \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 12 & 15 \end{bmatrix}$
6. If a matrix has 8 elements, what are the possible orders it can have ? What is it has 5 elements ?
7. (a) Construct a 2×3 matrix whose elements a_{ij} are given by :
   (i) $a_{ij} = = i . j$ (ii) $a_{ij} = 2i - j$
   (iii) $a_{ij} = i + j$ (iv) $a_{ij} = \frac{(i+j)^2}{2}$
   (b) Construct a 2×2 matrix whose elements a_{ij} are given by :
   (i) $\frac{(i+j)^2}{2}$ (ii) $a_{ij} = \frac{(i-j)^2}{2}$
   (iii) $a_{ij} = \frac{(i-2j)^2}{2}$ (iv) $a_{ij} = \frac{(2i-j)^2}{2}$
   (v) $a_{ij} = \frac{|2i-3j|}{2}$ (vi) $a_{ij} = \frac{|-3i+j|}{2}$
   (c) Construct a 3×4 matrix $A = [a_{ij}]$ whose elements a_{ij} are given by :
   $$a_{ij} = \frac{1}{2}|-3i+j|$$
8. For what values of x and y are the following matrices equal ?
   $$A = \begin{bmatrix} 2x+1 & 2y \\ 18z & y^2-5y \end{bmatrix}, B = \begin{bmatrix} x+3 & y^2+2 \\ 0 & -6 \end{bmatrix}$$
9. If $\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$, find x, y, z, w.

10. If $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ 4x+6 & a-1 & 0 \\ b-3 & 3b & z+2c \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ 2x & -3 & 2c-2 \\ 2b+4 & -21 & 0 \end{bmatrix}$

Obtain the values of a, b, c, x, y and z.

11. Find the values of a, b, c and d from the following equations :

$$\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$$

12. The sales figure of two car dealers during January 2007 showed that dealer A sold 5 deluxe, 3 premium and 4 standard cars, while dealer B sold 7 deluxe, 2 premium and 3 standard cars. Total sales over the 2 month period of January–February revealed that dealer A sold 8 deluxe 7 premium and 6 standard cars. In the same 2 month period, dealer B sold 10 deluxe, 5 premium and 7 standard cars. Write 2 × 3 matrices summarizing sales data for January and 2 month period for each dealer.

13. If $A = \begin{bmatrix} 123 \\ 456 \end{bmatrix}$, $B = \begin{bmatrix} 654 \\ 321 \end{bmatrix}$, then $A + B = \begin{bmatrix} 1+6 & 2+5 & 3+4 \\ 4+3 & 5+2 & 6+1 \end{bmatrix} = \begin{bmatrix} 7 & 7 & 7 \\ 7 & 7 & 7 \end{bmatrix}$

14. If $A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 7 & 2 \end{bmatrix}$, find 3A – 2B.

15. If A = diag(1 – 1 2) and B = diag (2 3 – 1), find A + B, 3A + 4B.

16. Simplify $\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

17. Find X and Y if $X + y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and $x - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

18. Find x, y, z, t if $2\begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$

19. Find a matrix X such that 2A + B + X = 0, where

$$A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$$

20. Find a matrix A such that 2A – 3B + 5C = 0, where

$$B = \begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix}$$

21. Solve the matrix equation $\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} - 2\begin{bmatrix} x \\ 2y \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$

22. Two farmers Ram Kishan and Gurcharan Singh cultivate only three varities of rice namely Basmati, Permal and Naura. The sale (in Rs) of these varities of rice by both the farmers in the month of September and October are given by the following matrices A and B.

September sales (in Rs.)

$$A = \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \begin{bmatrix} 10,000 & 20,000 & 30,000 \\ 50,000 & 30,000 & 10,000 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Ram Kishan} \\ \text{Gurcharan Singh} \end{array}$$

$$B = \begin{bmatrix} 5,000 & 10,000 & 6,000 \\ 20,000 & 10,000 & 10,000 \end{bmatrix} \begin{array}{l} \text{Ram Kishan} \\ \text{Gurcharan Singh} \end{array}$$

Find :

(i) What were the combined sales in September and October for each farmer in each variety.

(ii) What was the change in sales from September to October ?

(iii) If both farmers receive 2% profit on gross rupees sales, compute the profit for each farmer and for each variety sold in October.

23. Compute the following sums :

(i) $\begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ 1 & 3 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & 1 & 3 \\ 0 & 3 & 5 \\ -1 & 2 & 5 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 3 \\ 2 & 6 & 1 \\ 0 & -3 & 1 \end{bmatrix}$

24. If $A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 4 & 1 \end{bmatrix}, C = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix}$, find

(i) A + B and B + C

(ii) 2B + 3A and 3C – 4B

25. Find matrices X and Y, if $X + y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $X - Y \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$

26. If $X - Y = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $X + Y = \begin{bmatrix} 3 & 5 & 1 \\ -1 & 1 & 4 \\ 11 & 8 & 0 \end{bmatrix}$, find X and Y.

27. If $A = \begin{bmatrix} 9 & 1 \\ 7 & 8 \end{bmatrix}, B = \begin{bmatrix} 1 & 5 \\ 7 & 12 \end{bmatrix}$, find matrix C such that 5A + 3B + 2C is a null matrix.

28. If $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$, find the matrix C such that A + B + C is zero matrix.

29. Find the falue of λ, a non–zero scalar, if $\lambda\begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \end{bmatrix} + 2\begin{bmatrix} 1 & 2 & 3 \\ -1 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 10 \\ 4 & 2 & 14 \end{bmatrix}$

30. Find x, y, z and t, if

(i) $3\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2t \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+t & 3 \end{bmatrix}$

(ii) $2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 15 & 14 \end{bmatrix}$

31. In a certain city there are 30 colleges. Each college has 15 peons, 6 clerks, 1 typist and 1 section officer. Express the given information as a column matrix. Using scalar multiplication, find the total number of posts of each kind in all the colleges.

32. If $A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 4 & -3 \end{bmatrix}$, then find the A is a 3 × 3 matrix and B is a 3 × 2 matrix.

33. Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 3 & 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ -1 & 2 \\ 4 & -5 \end{bmatrix}$. Find AB and BA and show that $AB \neq BA$.

34. Matrix multiplication is not commutative in general.

35. Matrix multiplication is distributuve over matrix addition i.e.,

(i) A(B + C) = AB + AC

(ii) (A + B) C = AC + BC whenever both sides of euqality are defined.

36. If A is an m × n matrix, them $I_m A = A = A I_n$.

37. If A is m × n matrix and O is a null matrix, then

(i) $A_{m\times n} O_{n\times p} = O_{m\times p}$

(ii) $O_{p\times m}A_{m\times n} = O_{p\times n}$

i.e., the product of the matrix with a null matrix is always a null matrix.

38. If $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$, prove that $(A + B)^2 \neq A^2 + 2AB + B^2$

39. Prove that the product of matrices :

$$\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \text{and} \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

40. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A + B)^2 = A^2 + B^2$, find a and b.

41. If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, find x and y such that $(xI + yA^2) = A$.

42. If $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$, find the values of α for which $A^2 = B$.

43. Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}, C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$

Find a matrix D such that $CD - AB = 0$.

44. Let $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and I be the identity matrix of order 2. Show that I + A = (I – A)

$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

45. If $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$, find A.

46. Let $f(x) = x^2 - 5x + 6$. Find f(A) if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

47. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I_2 = O$.

48. Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ and $f(x) = x^2 - 4x + 7$. Show that (A) =). Use this result to find A^5.

49. Prove the following by the principle of mathematical induction :

If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ for every positive integer n.

50. If $A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then prove that

(i) $A_\alpha \cdot A_\beta = A_{\alpha+\beta}$

(ii) $(A)^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$, for every positive integer n.

51. If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, prove that

$$(aI + bA)^n = a^n I + na^{n-1} bA$$

where I is a unit matrix of order 2 and n is a positive integer.

52. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, then prove that

$$A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}$$ for every positive integer n.

53. Let A, B be two matrices such that they commute. Show that for any positive integer n.
(i) $AB^n = B^n A$
(ii) $(AB)^n = A^n B^n$

54. Under what conditions is the matrix equaion
$A^2 - B^2 = (A - B)(A + B)$ is true ?

55. If A is any $m \times n$ such that AB and BA are both defined show that B is an $n \times m$ matrix.

56. Give an example of three matrices A, B, C such that AB = AC but $B \neq C$.

57. Evaluate the following :

(i) $\left(\begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\right)\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 3 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix}\right)$

58. Show that $AB \neq BA$ in each of the following cases :

(i) $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & -1 & -1 \\ 3 & 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 9 & -4 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 10 & -4 & -1 \\ -11 & 5 & 0 \\ 9 & -5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 1 & 3 & 2 \end{bmatrix}$

59. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then show that $A^2 = B^2 = C^2 = I_2$

60. If $A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$, show that AB = A and BA = B.

61. For the following matrices verify the associativity of matrix multiplication i.e. (AB) C = A (BC) :

(i) $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 4 & 2 & 3 \\ 1 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

62. If w is a complex cube root of unity, show that

$$\left(\begin{bmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{bmatrix} + \begin{bmatrix} w & w^2 & 1 \\ w^2 & 1 & w \\ w & w^2 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ w \\ w^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

63. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, show that $A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

64. If $A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$, show that $A^2 = A$.

65. If $[1 \;\; 1 \;\; x] \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$, find x.

66. If $[1 \;\; -1 \;\; x] \begin{bmatrix} 0 & 1 & -3 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0$ find x.

67. Find the value of x for which the matrix product

$$\begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} -x & 14x & 7x \\ 0 & 1 & 0 \\ x & -4x & -2x \end{bmatrix}$$ equal an identity matrix.

68. A matrix X has a + b rows and a + 2 columns while the matrix Y has b + 1 rows and a + 3 columns. Both matrices XY and YX exist. Find a and b. Can you say XY and YX are of the same type ? Are they equal ?

69. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I_2 = O$

70. Show that the matrix $A = \begin{bmatrix} 5 & 3 \\ 12 & 7 \end{bmatrix}$ is a root of the equation $A^2 - 12A - I = O$.

71. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, then show that A is a root of the polynomial

$f(x) = x^3 - 6x^2 + 7x + 2$.

72. (i) If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, then prove that $A^2 - 4A - 5I = O$.

(ii) If $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix}$, and I is the identity matrix of order 3, show that

$A^3 + pI + qA + rA^2$.

73. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Use the principle of mathematical induction to show that

$$A^n = \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$$ for every positive integer n.

74. Give examples of matrices :
(i) A and B such that $AB \neq BA$.
(ii) A and B such that $AB = O$ but $A \neq 0$, $B \neq 0$.

(iii) A and B such that AB = O but BA ≠ O.
(iv) A, B and C such that AB = AC but B ≠ C, A ≠ 0.

75. If A and B are square matrices of the same order, explain, why in general :
(i) $(A + B)^2 \neq a^2 + 2AB + B^2$
(ii) $(A - B)^2 \neq A^2 - 2AB + B^2$
(iii) $(A + B)(A - B) \neq A^2 - B^2$

76. Find the matrix A such that
(i) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A = \begin{bmatrix} 3 & 3 & 5 \\ 1 & 0 & 1 \end{bmatrix}$
(ii) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$

77. If $A = \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix}$, find A^{16}.

78. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$ use this to find A^4.

79. Without using the concept of inverse of a matrix, find the matrix $\begin{bmatrix} x & y \\ z & u \end{bmatrix}$ such that

$$\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x & y \\ z & u \end{bmatrix} = \begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}$$

80. Solve the matrix equations :
(i) $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = 0$
(ii) $\begin{bmatrix} x & -5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = 0$

81. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$, find k such that $A^2 = kA - 2I_2$

82. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $f(x) = x^2 - 2x - 3$, show that $f(A) = 0$.

83. If $A = \begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}$, then prove by principle of mathematical induction that

$A^n = \begin{bmatrix} \cos n\theta & i\sin n\theta \\ i\sin n\theta & \cos n\theta \end{bmatrix}$ for all $n \notin N$.

84. In a legislative assembly election, a political group hired a public relations firm to promote its candidates in three ways : telephone, house calls and letters. The cost per contact (in paise) is given matrix A as

$$A = \begin{array}{c} \text{Cost per contact} \\ \begin{bmatrix} 40 \\ 100 \\ 50 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Telephone} \\ \text{House call} \\ \text{Letter} \end{array}$$

The number of contacts of each type made in two cities X and Y is given in matrix B as

$$B = \begin{array}{c} \begin{array}{ccc} \text{Telephone} & \text{House call} & \text{Letter} \end{array} \\ \begin{bmatrix} 1000 & 500 & 5000 \\ 3000 & 1000 & 10000 \end{bmatrix} \end{array} \begin{array}{l} \\ \to X \\ \to Y \end{array}$$

Find the total amount spent by the group in the two cities X and Y.

85. A trust fund has Rs. 30000 that must be invested in two different types of bonds. The first bond pays 5% interest per year, and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide Rs. 30000 among the two types of bonds. If the trust fund must obtain an annual total interest of

(i) Rs. 1800 (ii) Rs. 2000

86. For any two matrices A and B of the same order,

$(A + B)^T = A^T B^T$

87. If A and B are two matrices such that AB is d efined, then

$(AB)^T = B^T A^T$

88. If A, B, C are three m atric es confirmable for the products (AB) C and A(BC), then

$(ABC)^T = C^T B^T A^T$.

89. If $A = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and $B = [-2 \ \ -1 \ \ -4]$, verify that $(AB)^T = B^T A^T$

90. If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then find the values of θ satisfying the equation $A^T + A = I_2$.

91. Find the valuesof x, y, z ifthe matrix$A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ sagisfy the equation $A^T A = I_3$.

92. If $A = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$ and $B = [1 \ \ 0 \ \ 4]$, verify that $(AB)^T = B^T A^T$

93. If $A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$, $B = [1\ 3\ \ -6]$, verify that $(AB)^T = B^T A^T$

94. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, they verify that $A^TA = I_2$

95. If $A = \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}$, verify that $A^T A = I_2$.

96. Show that the element on the ma in diag on al of a skew–symmetic matrix are all zero.

97. Prove that every square matrix c an be uniquely expressed as the sumof a symmetric matrix and a skew–symmetricmatrix.

98. Show that the matrix $B^T AB$ is symmetric according as A is symmetric or skew–symmetric.

99. Show that all positiv e integral powers of a symmetric matrix are symmetric.

100. Let A and B be symmetric matrices of the same order. Then, show that

(i) A +B is a symmetric matrix.

(ii) AB – BA is a skew–symmetric matrix.

(iii) AB + BA is a symmetric matrix.

101. Express the matrix $A = \begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 5 \end{bmatrix}$ as the sum of a symmetric and a skew–symmetric matrix.

102. A matrix which is both symmetric as well as skew–symmetric is a null matrix.

103. Express the matrix $A = \begin{bmatrix} 4 & 2 & -1 \\ 3 & 5 & 7 \\ 1 & -2 & 1 \end{bmatrix}$ as the sum of a symmetric and a skew–symmetric matrix.

104. Express the f oll owing matrix as the sum of a symmetric and skew–symmetric matrix and verify your result : $\begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

EXERCISE-2

Answer each of the following questions in one word or one sentence or as per exact requirement of the questions :

1. Write matrix Asatisfying $A + \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ -3 & 6 \end{bmatrix}$.

2. If $A = [a_{ij}]$ is a skew–symetric matrix, then write the vaue of $\sum_i a_{ij}$.
3. If B is a symmetric matrix, write whether the matrix $AB A^T$ is symmetric or skew–symmetric.
4. If A is a symmetric matrix and $n \in N$, write whether A^n is symmetric or skew–symmetric or neither or these two.
5. If A and B are symmetric matrices of the same order, write whether AB – BA is symmetric or skew–symmetric or neither of the two.
6. If $\begin{bmatrix} x+3 & 4 \\ y-4 & x+y \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 9 \end{bmatrix}$, find x and y.
7. Find the value ofy, if $\begin{bmatrix} x-y & 2 \\ x & 5 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$.
8. Find the vaue of x if $\begin{bmatrix} 3x+y & -y \\ 2y-x & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -5 & 3 \end{bmatrix}$.
9. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find $A + A^T$.
10. If $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ k & 23 \end{bmatrix}$, then write the value of k.
11. If $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ is written B +C, where B is a symmetric matrix and C is a skew–symmetric matrix, then B is euqal to.
12. If A is 2×3 matrix and B is a matrix such that $A^T B$ and BA^T both are d efined, then what is the order of B ?

EXERCISE-3

1. If AB = A and BA =B, where A and B are square matrices, then
   (a) $B^2 = B$ and $A^2 = A$ (b) $B^2 \neq B$ and A^2 A (c) $A^2 \neq A$, $B^2 = B$ (d) $A^2 \neq A$, $B^2 \neq B$
2. Let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$, then A^n is equal to

   (a) $\begin{bmatrix} a^n & 0 & 0 \\ 0 & a^n & 0 \\ 0 & 0 & a \end{bmatrix}$ (b) $\begin{bmatrix} a^n & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ (c) $\begin{bmatrix} a^n & 0 & 0 \\ 0 & a^n & 0 \\ 0 & 0 & a^n \end{bmatrix}$ (d) $\begin{bmatrix} na & 0 & 0 \\ 0 & na & 0 \\ 0 & 0 & na \end{bmatrix}$

3. If $A = \begin{bmatrix} 1 & 2 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $AB = I_3$, then x + y equals

(a) 0 (b) –1 (c) 2 (d) none of these

4. If a matrix Ais both symmetric and skew–symmetric, then

(a) A^k (b) I – A (c) I (d) 3A

5. If $A = \begin{bmatrix} 5 & x \\ y & 0 \end{bmatrix}$ and $A = A^T$, then

(a) x = 0, y = 5 (b) x + y = 5 (c) x = y (d) none of these

6. If $A = [a_{ij}]$ is a square matrix of even ordersuch that $a_{ij} = i^2 - j^2$, then

(a) A is a skew–symmetric matrix and | A | = 0

(b) A is symmetric matrix and | A | is a square

(c) A is symmetric matrix and | A |= 0

(d) none of these

7. If $A = \begin{bmatrix} 2 & 0 & -3 \\ 4 & 3 & 1 \\ -5 & 7 & 2 \end{bmatrix}$ is expressed as the sum of a symmetric and skew–symmetric matrix, then the symmetric matrix is

(a) $\begin{bmatrix} 2 & 2 & -4 \\ 2 & 3 & 4 \\ -4 & 4 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 4 & -5 \\ 0 & 3 & 7 \\ -3 & 1 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 4 & 4 & -8 \\ 4 & 6 & 8 \\ -8 & 8 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. The number of allpossible matrices of order 3 × 3with each entry 0 or 1 is

(a) 27 (b) 18 (c) 81 (d) 512

9. If $A = \begin{bmatrix} 0 & 2 \\ 3 & -4 \end{bmatrix}$ and $kA = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$,then the values of k, a, b are respectively

(a) –7, –12, –18 (b) –6 , 4, 9 (c) –6, –4, –9 (d) –6, 12, 18

10. The trace of the matrix $A = \begin{bmatrix} 1 & -5 & 7 \\ 0 & 7 & 9 \\ 11 & 8 & 9 \end{bmatrix}$ is

(a) 17 (b) 25 (c) 3 (d) 12